On the Calculation of Vector Mesons in Functional Nonlinear Spinor Theory

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Nonlinear spinor theory is fomulated in functional space. An eigenvalue equation for mesons is derived. The group theoretical reduction of this equation is performed, especially the angular momentum decomposition. For vector mesons it is solved in first Fredholm approximation. A solution corresponding to a physical particle is found contrary to earlier calculations. The calculated mass has the correct order of magnitude.

1. Introduction

The theory of high energy phenomena should be formulated in the framework of a unified relativistic quantum field theory. HEISENBERG and coworkers have proposed a nonlinear spinor equation [1, 2], which should principally be able to describe the whole spectrum of elementary particles. Heisenberg's theory is formulated in terms of field operators, but until today no solution procedure for operator equations of this type is known. Moreover it is quite unclear what a field operator does mean at all. Nevertheless, starting from this rather dubious concept it is possible to carry out calculations proceeding as follows. Postulating the existence of a cyclic base vector system generated by the field operators, one is able to derive an infinite set of coupled differential equations for the projections of a certain state on the cyclic base vectors system, that is for the set $\tau_n{}^a(x_1 \ldots x_n) = \langle 0 | T\,\psi(x_1) \ldots \psi(x_n) | a \rangle$, $n = 1, \ldots, \infty$. Such an infinite system can be written as a simple functional equation which is usually regarded as a formal remedy, making difficult and tricky manipulations easier to handle.

Next one can try to calculate the τ-functions and the corresponding vacuum expectation values in some approximation. Then principally all physical information should be available. But this approach has two serious drawbacks: First it is impossible to calculate S-matrix without refering to the states. This stems from the fact that in a unified theory of elementary particles each particle is a relativistic cluster, that means it has an internal structure. No particle is more elementary than the other, although they are not equivalent. Therefore the Green functions do not suffice to determine the S-matrix. Second, the set of all $\tau_n{}^a$-functions represents in some sense the state $|a\rangle$, that is the τ-functions can serve as expansion coefficients in a series expansion for $|a\rangle$. Such an expansion refers explicitly to the cyclic basis system generated by the field operators, that is to quantities without definite meaning.

The whole set of τ-functions is supposed to be in one-to-one correspondence to the set of states, similar to infinite sequences of numbers which are under certain conditions in one-to-one correspondence to the space of square integrable functions. Lead from physical intuition it is possable to justify the approximation schemes used so far, but from a fundamental point of view the situation is quite unsatisfactory. There are rather some recipes than a theory.

In our opinion a unified theory of elementary particles should be formulated in a space $\mathcal{H}$ of state vectors provided with an inner product which preserves probability and gives a clear-cut meaning to approximation schemes. The global observables should be given as the infinitesimal generators of the invariance group of the theory, represented as Hermitian operators on $\mathcal{H}$. Furthermore there should be a dynamical equation for the calculation of bound states and masses, that is the relativistic counterpart to the Schrödinger equation. Then the S-matrix would be given by the scalar product between in- and out-states of relativistic clusters.

A possible realization of this program is provided by the functional version of nonlinear spinor theory, developed by STUMPF and coworkers. We will not go into particulares, they can be found in a detailed review article [3] and the literature cited there. Here we will restrict ourselves to some general remarks.

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The main idea is to regard the functional equation for the τ-functions not simply as a shorthand-writing, but together with the physical boundary conditions as the definition of the theory. The space of solutions of the functional equation is equipped with a scalar product which turns it into a Hilbert space. This allows defining the S-matrix and provides us with a probability interpretation of the theory. This is in perfect analogy to ordinary quantum mechanics: the functional equation corresponds to the Schrödinger equation which itself can be regarded as a functional equation of a very simple type.

One complication we have ignored so far. To make the theory finite, Heisenberg uses a special noncanonical quantization procedure which induces an indefinite metric in the state space [1]. There will be solutions of the fundamental functional equation which do not correspond to physical particles. These "ghosts" must have negative or zero norms. Therefore the functional space to be dealt with cannot be a Hilbert space, but must be a larger space which contains the Hilbert space of physical states [4]. Principally, one cannot decide a priori whether a solution corresponds to a physical particle or not. This must be tested by scattering calculations. The identification of physical amplitudes is part of the dynamical problem in functional nonlinear spinor theory.

In this paper we deal with the bound state problem for mesons, especially for vector mesons. Meson calculations have been carried out already in the original version of nonlinear spinor theory in the lowest order of the so-called new Tamm-Dancoff-approximation by several authors [1, 5]. In this approximation the solution for particles with spin 1 correspond to the above mentioned ghosts and not to the physical vector mesons. The second order Tamm-Dancoff-approximation has been investigated in [6, 7]. In [6] the second order equation for scalar mesons has been reduced to a multiple of the first order equation. Also in [7] only scalar mesons were considered, because the angular momentum decomposition was not performed. So until now there exists no calculation of the masses of physical vector particles in the framework of nonlinear spinor theory. In the next section we give the fundamentals of functional nonlinear spinor theory and derive the meson eigenvalue equation. In lowest approximation it corresponds to a systematic version of the second order Tamm-Dancoff-approximation. In Sectins 3 and 4 we carry out the group theoretical analysis, especially the angular momentum decomposition. The resulting equation is a Bethe-Salpeter equation [8] of a very complicated structure. The solution of Bethe-Salpeter equations of this complexity has not been discussed in the literature until now. To get a first hint, the eigenvalues are calculated in Fredholm approximation in Section 5. Technical details are discussed in the Appendices.

2. Fundamentals

a) The basic equation of functional nonlinear spinor theory is [3]:

$$\{D_\alpha{}^\beta(x)\,\partial_\beta(x) - V_\alpha{}^{\beta\gamma\delta}[\partial_\beta(x)\,\partial_\gamma(x)\,\partial_\delta(x) - 3F_{\beta\gamma}(a)\,\partial_\delta(x)] - i\varrho_0 j_\alpha(x)\}\,|\,\mathfrak{T}_a(j)\rangle = 0\,. \quad (2.1)$$

In (2.1) we have introduced the summation convention for all indices and variables. $D_\alpha{}^\beta(x)$ is the Hermitian Dirac operator [9] and $V_\alpha{}^{\beta\gamma\delta}$ is the vertex operator of DÜRR and WAGNER [5]. The subtraction of $3F(0)$ corresponds to the normal ordering of the vertex expression in the operator equation. $j_\alpha(x)$ and $\partial_\alpha(x)$ are functional operators with the anticommutation rules:

$$[j_\alpha(x), j_\beta(y)]_+ = [\partial_\alpha(x), \partial_\beta(y)]_+ = 0\,,$$
$$[j_\alpha(x), \partial^\beta(y)]_+ = \delta_\alpha{}^\beta\,\delta(x-y)\,. \quad (2.2)$$

A state $|a\rangle$ and the corresponding state functional $|\mathfrak{T}_a(j)\rangle$ are characterized by quantum numbers. They are given by the eigenvalues of a maximal set of simultaneous diagonalizable infinitesimal generators of the underlying symmetry group, that is the Poincaré group and inner symmetry groups. The choice of the maximal set is to some extent arbitrary. We choose the linear momentum $\mathbf{P}_\mu$, its square $\mathbf{P}^2 = m^2$, angular momentum $\mathbf{\Gamma}$, and its projection $\mathbf{\Gamma}_3$. Hence $|\mathfrak{T}_a(j)\rangle$ has to fulfill the following subsidiary conditions:

$$\begin{aligned} &\text{a)} \quad (\mathbf{P}_\mu - J_\mu)\,|\mathfrak{T}_a(j)\rangle = 0\,,\\ &\text{b)} \quad (\mathbf{P}^2 - m^2)\,|\mathfrak{T}_a(j)\rangle = 0\,,\\ &\text{c)} \quad (\mathbf{\Gamma}^\mu\mathbf{\Gamma}_\mu - j(j+1))\,|\mathfrak{T}_a(j)\rangle = 0\,,\\ &\text{d)} \quad (\mathbf{\Gamma}_3 - j_3)\,|\mathfrak{T}_a(j)\rangle = 0\,. \end{aligned} \quad (2.3)$$

Isospin invariance and baryon number conservation require additionally:

$$\begin{aligned} &\text{a)} \quad (\mathbf{T}^2 - I(I+1))\,|\mathfrak{T}_a(j)\rangle = 0\,,\\ &\text{b)} \quad (\mathbf{B} - b)\,|\mathfrak{T}_a(j)\rangle = 0\,. \end{aligned} \quad (2.4)$$

The operators $\mathbf{P}_\mu$, $\mathbf{\Gamma}_\mu$, $\mathbf{T}_i$ and $\mathbf{B}$ are the representation of linear momentum, angular momentum, iso-

spin, and baryon number in functional space. An operator **A** in functional space corresponding to an operator $A(x)$ in ordinary spinor space is defined by:

$$\mathbf{A} = \int \mathrm{d}x\, j_\alpha(x)\, A(x)\, \vartheta^\alpha(x). \tag{2.5}$$

For example the operator of linear momentum reads in functional space:

$$\mathbf{P}_\mu = -i \int \mathrm{d}x\, j_\alpha(x)\, \frac{\partial}{\partial x^\mu}\, \vartheta^\alpha(x). \tag{2.6}$$

For further details see [10–12].

Originally the Eqs. (2.1), (2.3), (2.4) were derived from Heisenberg's nonlinear spinor equation by assuming a series expansion for $|\, \mathfrak{T}_a(j)\,\rangle$:

$$|\, \mathfrak{T}_a(j)\,\rangle = \sum_{n=0}^{\infty} \frac{i^n}{n!}\, \tau_n^a \begin{pmatrix} x_1 \dots x_n \\ \alpha_1 \dots \alpha_n \end{pmatrix} j^{\alpha_1}(x_1) \dots j^{\alpha_n}(x_n)\, |\, \varphi_0\rangle$$

$$\tau_n{}^a \begin{pmatrix} x_1 \dots x_n \\ \alpha_1 \dots \alpha_n \end{pmatrix} := \langle 0\,|\, T\, \psi_{\alpha_1}(x_1) \dots \psi_a(x_n)\,|\, a\rangle\,. \tag{2.7}$$

ψ is an Hermitian-Weyl spinor field operator [5], and $|\, \varphi_0\rangle$ serves as a functional groundstate, defined by:

$$\vartheta_a(x)\,|\, \varphi_0\rangle = 0\,. \tag{2.8}$$

We emphasize that our basic Eqs. (2.1), (2.3), (2.4) are assumed to be independent from an expansion like (2.7). Still (2.7) can be used to get approximative solutions of functional equations (and we will do so), but this is not mandatory.

b) In Eq. (2.1) the quantization enters only through the term $i\varrho_0 j_\alpha(x)$ which vanishes in the case of noncanonical quantization. The information about the quantization is then introduced by the two-point function

$$F_{\alpha\beta}(x-y) := \langle 0\,|\, T\, \psi_\alpha(x)\, \psi_\beta(y)\,|\, 0\rangle$$

with help of the normal transform

$$|\, \mathfrak{T}_a(j)\,\rangle = \exp\{-\tfrac{1}{2} j_\alpha(x)\, F^{\alpha\beta}(x-y)\, j_\beta(y)\}\,|\, \Phi_a(j)\,\rangle\,. \tag{2.9}$$

In the case of free fields (2.9) is the functional version of Wick's theorem [13]. Substitution of (2.9) into (2.1) gives:

$$\{\vartheta_\varkappa(x) - G_{\varkappa\alpha}(x,y)\, V^{\alpha\beta\gamma\delta}[d_\beta(y)\, d_\gamma(y)\, d_\delta(y) - 3\, F_{\beta\gamma}(0)\, d_\delta(y)] - F^1_{\varkappa\alpha}(x-y)\, j^\alpha(y)\}\,|\, \Phi_a(j)\,\rangle \tag{2.10}$$

with $$d_\varkappa(x) := \vartheta_\varkappa(x) - F_{\varkappa\alpha}(x-y)\, j^\alpha(y) \tag{2.11}$$

and

$$F^1_{\varkappa\alpha}(x-y) = F_{\varkappa\alpha}(x-y) + i\varrho_0\, G_{\varkappa\alpha}(x-y). \tag{2.12}$$

In (2.11) we have inverted the differential operator $D_{\alpha\beta}(x)$ by means of the causal Green function $G_{\varkappa\alpha}(x-y)$. The possible inhomogeneous term vanishes for the bound state problem of mass $m \neq 0$. The conditions (2.3), (2.4) apply unchanged to $|\, \Phi_a(j)\,\rangle$.

In the next step (2.10) has to be averaged in the free variable x in order to get a symmetrical set of equation for the τ-functions. If this is not done, the known difficulties arise that the iteration of different coordinates leads to structural different equations [6]. This averaging can be done in different ways. In [14] an indefinite functional integral was used which reproduces exactly the symmetrical Tamm-Dancoff-approximation. We follow the way proposed in [15] and apply $\int \mathrm{d}x\, j^\varkappa(x)\, P_\mu(x)$ to (2.10):

$$j^\alpha(x)\, P_\mu(x)\, \vartheta_\alpha(x)\,|\, \Phi_a(j)\,\rangle = \mathbf{P}_\mu\,|\, \Phi_a(j)\,\rangle = \mathfrak{D}_\mu(j,d)\,|\, \Phi_a(j)\,\rangle \tag{2.13}$$

with the dynamical operator

$$\mathfrak{D}_\mu(j,d) = j^\varkappa(x)\, P_\mu(x)\, G_{\varkappa\alpha}(x,y)\, V^{\alpha\beta\gamma\varrho} \cdot [d_\beta(y)\, d_\gamma(y)\, d_\varrho(y) - 3\, F(0)\, d_\varrho(y)]\,. \tag{2.14}$$

Squaring of both sides of (2.13) then yields:

$$m^2\,|\, \Phi_a(j)\,\rangle = \mathfrak{D}_\mu\, \mathfrak{D}^\mu\,|\, \Phi_a(j)\,\rangle\,. \tag{2.15}$$

(2.15) represents the relativistic analogon to the Schrödinger equation. The difference lies in the fact that Eq. (2.15) is defined on a space of functionals, whereas the Schrödinger equation is usually defined on the space of the very special functionals $f(x)$, with f square integrable. But principally this difference is insignificant. The aim is to solve (2.14) by direct methods, but these do not exist yet.

Therefore we look for an approximate solution of (2.15). We make the ansatz:

$$|\, \Phi_a(j)\,\rangle = \sum_{n=0}^{\infty} \frac{i^n}{n!}\, \varphi_n \begin{pmatrix} x_1 \dots x_n \\ \alpha_1 \dots \alpha_n \end{pmatrix} \cdot j^{\alpha_1}(x_1) \dots j^{\alpha_n}(x_n)\,|\, \varphi_0\rangle\,. \tag{2.16}$$

Due to grouptheoretical reasons φ_n vanishes for $n < \varrho$, where ϱ depends on the state $|\, a\rangle$. For the nucleon e. g. we have $\varrho = 1$, and $\varrho = 2$ for the mesons, regarded as a boundstate of fermion-antifermion systems. By dividing the functional into two parts

$$|\, \Phi_a(j)\,\rangle = \frac{i^\varrho}{\varrho!}\, \varphi_\varrho \begin{pmatrix} x_1 \dots x_\varrho \\ \alpha_1 \dots \alpha_\varrho \end{pmatrix} j^{\alpha_1}(x_1) \dots j^{\alpha_\varrho}(x_\varrho)\,|\, \varphi_0\rangle + \sum_{n=\varrho+1}^{\infty} \frac{i^n}{n!}\, \varphi_n \begin{pmatrix} x_1 \dots x_n \\ \alpha_1 \dots \alpha_n \end{pmatrix} j^{\alpha_1}(x_1) \dots j^{\alpha_n}(x_n)\,|\, \varphi_0\rangle = |\, \Phi_\varrho(j)\,\rangle + |\, \Phi_{\mathrm{rest}}(j)\,\rangle \tag{2.17}$$

one can derive a functional equation for $|\Phi_\varrho\rangle$ alone (l. c. [15]):

$$m^2 |\Phi_\varrho(j)\rangle - \mathbf{\Pi}_\varrho \mathfrak{D}_\mu \mathfrak{D}^\mu \left[1 - \frac{1}{m^2} \hat{\mathbf{\Pi}}_\varrho \mathfrak{D}_\mu \mathfrak{D}^\mu\right]^{-1} \cdot |\Phi_\varrho\rangle = 0 . \quad (2.18)$$

$\mathbf{\Pi}_\varrho$ and $\hat{\mathbf{\Pi}}_\varrho$ are functional projection operators defined by

$$\mathbf{\Pi}_\varrho |\Phi(j)\rangle = |\Phi_\varrho(j)\rangle ,$$
$$\hat{\mathbf{\Pi}}_\varrho = \sum_{n=\varrho+1}^{\infty} \mathbf{\Pi}_n . \quad (2.19)$$

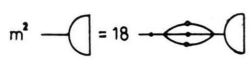
For the inverse operator in (2.18) we assume a Neumann series expansion and approximate it by the first term yielding the equation:

$$\mathbf{\Pi}_\varrho (m^2 - \mathfrak{D}_\mu \mathfrak{D}^\mu) |\Phi_\varrho\rangle = 0 . \quad (2.20)$$

From (2.20) with $\varrho = 2$ we will calculate the meson masses, and with $\varrho = 1$ the coupling constant giving the mass scale. With our averaging operator the occurring Feynman integrals are more involved compared to the integral average in [14], but the structure of the equations for higher φ-functions is drastically simplified because the last term in (2.10) vanishes identically due to the subsidiary condition for $F_{\alpha\beta}(x-y)$.

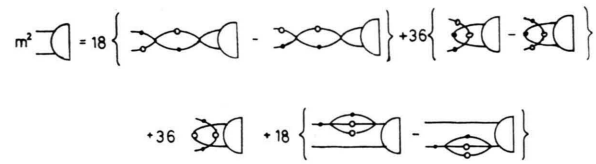
3. The Meson Eigenvalue Equation

a) The functional eigenvalue Eq. (2.20) can be transformed into an equation in coordinate space applying $\langle\varphi_0|\partial_\alpha(x)$ for $\varrho = 1$ and $\langle\varphi_0|\partial_\alpha(x)\,\partial_\beta(y)$ for $\varrho = 2$, respectively. We write the resulting equations immediately in graphical notation in order to demonstrate their structure. For $\varrho = 1$ we get:



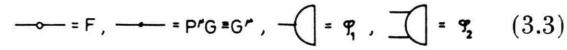
(3.1)

and for $\varrho = 2$:



(3.2)

with the following notation:



(3.3)

With Eq. (3.1) we wil deal in Section 5, here we will consider only Equation (3.2). Essentially it reproduces the result one gets from the Tamm-Dancoff-approximation, but in contrary to this our derivation is systematic and free from ambiguities. Especially we have no possibility of choosing special graphs as in [7].

(3.2) is a Bethe-Salpeter type equation of a very complicated structure: It is a spinor-spinor equation with unequal masses in the bubble approximation with normal and exchange interaction (the last fact cannot be seen in the Hermitean representation). Equations of such a difficult type have not been investigated in the literature so far. It is not our aim in this paper to develop a complete theory for such equations. We only want to get some first insight. Therefore we neglect some of the graphs in (3.2). First, the last two graphs are neglected since they will not effect the binding very much. Their influence could be taken into account by dressing the external Green function to a finite mass $\varkappa$.

Second, we neglect the fifth graph. For the group-theoretical reduction this is only a matter of notational convenience, since its algebraic structure is identical to the structure of the other nonlocal graphs.

The remaining equation reads in the momentum space in the center of mass system with the center of mass momentum J:

$$m^2 \varphi_{\alpha\beta}(q) = \frac{18}{(2\pi)^8} [G^\mu_{\alpha\delta}(q_-) F_{\beta\gamma}(q_+) - G^\mu_{\beta\gamma}(q_+) F_{\alpha\delta}(q_-)] \int \mathrm{d}k\, F_{\varkappa\gamma'}(k)$$
$$G_{\mu,\sigma\varepsilon}(k+J)\, V^{\varepsilon\delta\gamma\varkappa} V^{\gamma'\sigma\varrho\eta} \int \mathrm{d}p\, \varphi_{\eta\varrho}(p) + \frac{36}{(2\pi)^8} G^\mu_{\alpha\gamma'}(q_-) F_{\sigma\beta}(q_+) V^{\gamma\delta\varepsilon\eta} V^{\gamma'\varkappa\sigma\varrho}$$
$$\cdot \int \mathrm{d}p\, \mathrm{d}k\, G_{\mu,\varepsilon\gamma'}(k+q-p)\, F_{\varkappa\delta}(k)\, \varphi_{\eta\varrho}(p) - \frac{36}{(2\pi)^8} G^\mu_{\beta\gamma'}(q_+) F_{\sigma\alpha}(q_-) V^{\gamma\delta\varepsilon\eta} V^{\gamma'\varkappa\sigma\varrho} \quad (3.4)$$
$$\cdot \int \mathrm{d}p\, \mathrm{d}k\, G_{\mu,\varepsilon\gamma'}(k+p+q)\, F_{\varkappa\delta}(k)\, \varphi_{\eta\varrho}(p)$$

with $q_\pm = q \pm \frac{J}{2}$.

In (3.4) we used the following definitions:

$$F_{\alpha\beta}(p) = -i(\lambda_2' \bar{\sigma}_\mu')_{\alpha\beta} \int_0^\infty dm^2\, \varrho(m^2) \cdot m^4\, p^\mu/(p^2)^2\,(p^2-m^2) \tag{3.5}$$

is the dipole regularized two point function. In the practical calculations the spectral function $\varrho(m^2)$ will be replaced by a δ-function at the nucleon mass:

$$\varrho(m^2) = \delta(m^2-\varkappa^2)\,;\; G_{\alpha\beta}(p) = -\,(\lambda_2' \bar{\sigma}_\mu')_{\alpha\beta} \cdot p^\mu/p^2 \tag{3.6), (3.7}$$

is the free particle Green function, an the completely antisymmetrical vertex operator reads explicitly:

$$V_{\alpha\beta\gamma\delta} = \frac{l^2}{12}\,\{3\,\gamma^{'\mu}_{\alpha\beta}\,\sigma'_{\mu,\gamma\delta} + (\lambda_2' \boldsymbol{\tau}'\sigma'^\mu)_{\alpha\beta}\,(\lambda_2' \boldsymbol{\tau}'\sigma_\mu')_{\gamma\delta} + 2(\lambda_1'' \boldsymbol{\tau}')_{\alpha\beta}\,(\lambda_1'' \boldsymbol{\tau}')_{\gamma\delta} + 2(\lambda_3'' \boldsymbol{\tau}')_{\alpha\beta}\,(\lambda_3'' \boldsymbol{\tau}')_{\gamma\delta}\}\,. \tag{3.8}$$

The matrices λ', λ'', $\boldsymbol{\tau}'$ and σ' are direct products of Pauli matrices in isospin-, spin-, and the so called λ-space. So every index α is meant as a triple $(\alpha_1\,\alpha_2\,\alpha_3)$

$$\begin{aligned}
\sigma'^\mu &= (-i\lambda_2 I I,\; -i\lambda_2 I \sigma_1,\; i I I \sigma_2,\; -i\lambda_2 I \sigma_3)\\
\tau_i' &= (-i\lambda_2 \tau_1 I,\; i I \tau_2 I,\; -i\lambda_2 \tau_3 I)\\
\lambda_i' &= (i\lambda_1 I \sigma_2,\; -i\lambda_2 I I,\; i\lambda_3 I \sigma_2)\\
\lambda_i'' &= (-\lambda_1 \tau_2 \sigma_2,\; \lambda_2 I I,\; -\lambda_3 \tau_2 \sigma_2)\,.
\end{aligned} \tag{3.9}$$

The occurrence of the additional λ-space index stems from the fact that we are working in the Hermitian representation of the theory. $\bar{\sigma}'^\mu$ is defined by $\bar{\sigma}'^\mu := \sigma_\mu'$.

In (3.4) the integration over k can be performed by the standard techniques for the evaluation of Feynman integrals [16], yielding:

$$\begin{aligned}
\int dp\, F_{\alpha\beta}(p_-)\, G^\beta_{\gamma\delta}(p_+) &= (\lambda_2' \bar{\sigma}_\mu')_{\alpha\beta}\,(\lambda_2' \bar{\sigma}_\nu')_{\gamma\delta}\, M^{\mu\nu\lambda};\; M^{\mu\nu\lambda}(\lambda) = -i\varkappa^4 \int dp\, \frac{p_-^\mu\, p_+^\nu\, p_+^\lambda}{(p_-^2)^2\,(p_-^2-\varkappa^2)\,p_+^2}\\
&= \pi^2\{J^\mu J^\nu J^\lambda A^{(1)}(\lambda) + (g^{\mu\nu} J^\lambda + g^{\mu\lambda} J^\nu + g^{\nu\lambda} J^\mu)\; A^{(2)}(\lambda) + g^{\nu\lambda} J^\mu A^{(3)}(\lambda)\}
\end{aligned} \tag{3.10}$$

with $\lambda = J^2/\varkappa^2$ and the functions

$$\begin{aligned}
A^{(1)}(\lambda) &= \left(-\frac{1}{12} + \frac{1}{2\lambda^2} - \frac{2}{3\lambda^3} + \frac{1}{4\lambda^4}\right)\ln(1-\lambda) + \frac{1}{12}\ln\lambda - \frac{1}{12\lambda} - \frac{13}{24\lambda^2} + \frac{1}{4\lambda^3}\\
A^{(2)}(\lambda) &= -\frac{\varkappa^2}{2}\left[\left(\frac{\lambda}{12} - \frac{1}{3} + \frac{1}{2\lambda} - \frac{1}{3\lambda^2} + \frac{1}{12\lambda^3}\right)\ln(1-\lambda) + \left(\frac{1}{3} - \frac{\lambda}{12}\right)\ln\lambda + \frac{1}{12} - \frac{7}{24\lambda} + \frac{1}{12\lambda^2}\right]\\
A^{(3)}(\lambda) &= \frac{\varkappa^2}{2}\left[\left(\frac{\lambda}{6} - \frac{1}{2} + \frac{1}{2\lambda} - \frac{1}{6\lambda^2}\right)\ln(1-\lambda) + \left(\frac{1}{2} - \frac{\lambda}{6}\right)\ln\lambda + \frac{1}{6} - \frac{1}{6\lambda}\right]
\end{aligned} \tag{3.11}$$

These functions are not independent from another, but are related by

$$J^2 A^{(1)} = -6A^{(2)} - 4A^{(3)}\,. \tag{3.12}$$

b) (3.7) is a set of 64 coupled integral equations for 64 unknown functions. The number of components can be reduced considerably by group theoretical methods. The reduction of spinor-spinor Bethe-Salpeter equations has been discussed by various authors [17, 18], but only for Dirac spinors and not for the Weyl case. So we repeat here the discussion for the Weyl spinors. First we investigate the dependence of the meson wavefunction $\varphi_{\alpha\beta}(q)$ on baryon number, isospin, and spin.

In the Hermitian formulation of the theory one gets an unphysical mixture between baryon and antibaryon states. In a group theoretical language this means that we use a two dimensional representation of the baryon number gauge group in λ-space in a basis, where it does not decompose. In this representation, the baryon number operator has the form (l. c. [19]):

$$\beta = -\lambda_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \tag{3.13}$$

with eigenvectors

$$\lambda^{\mathrm{F}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}\quad \lambda^{\mathrm{A}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \tag{3.14}$$

"F" and "A" stand for fermion and antifermion, respectively. The eigenvectors of a two particle state are given by all direct products $\Lambda = \lambda^{\mathrm{A}} \otimes \lambda^{\mathrm{B}}$ of the vectors (3.14). They fulfill the orthonormality re-

lations

$$\Lambda^{AB}\,\Lambda^{A'B'} = \delta_{AB,\,A'B'}\,, \tag{3.15}$$

AB and A′B′ run through all possible combinations of F and A.

To investigate the isospin dependence of $\varphi_{\alpha\beta}$, we use the condition (2.4) for the normal transformed functional $|\,\Phi_a(j)\,\rangle$. Using the power series expansion (2.16) for $|\,\Phi_a(j)\,\rangle$ one gets by projection for $\varphi_{\alpha\beta}$:

$$I(I+1)\,\varphi_{\alpha\beta} = -\tfrac{1}{4}\,(\boldsymbol{\tau}'\otimes 1+1\otimes\boldsymbol{\tau}')^{2}_{\alpha\alpha'\beta\beta'}\,\varphi_{\alpha'\beta'}\,. \tag{3.16}$$

(3.16) is the isospin condition in λ-space. The isospin condition for a particle-antiparticle bound-state, that is for baryon number 0, results when projecting with Λ^{+FA}:

$$\begin{aligned} I(I+1)\,\varphi^{FA}_{\alpha\beta} &= \tfrac{1}{4}\,(\boldsymbol{\tau}\otimes 1+1\otimes\boldsymbol{\tau}^{A})^{2}_{\alpha\alpha'\beta\beta'}\,\varphi^{FA}_{\alpha'\beta'} \\ &= \tfrac{1}{2}\,(3+\boldsymbol{\tau}\otimes\boldsymbol{\tau}^{A})_{\alpha\alpha',\,\beta\beta'}\,\varphi^{FA}_{\alpha'\beta'} \end{aligned} \tag{3.17}$$

with $$\boldsymbol{\tau}^{A} = -\,\boldsymbol{\tau}^{T}$$

The possible values for I are 1 and 0. With help of the τ-algebra one easily verifies:

The eigenvectors for $I=0$ is $\tau_0 = 1$ (singlet)

and for $I=1$ are (τ_1,τ_2,τ_3) (triplet).

The angular momentum dependence of $\varphi_{\alpha\beta}$ follows from the condition (2.3c). The total angular momentum of a two particle system is given by the sum of the individual spins of the particles and their relative orbital angular momentum. Hence we are concerned with the coupling of three angular momenta. We use the *LS*-coupling scheme, that is we first couple the two individual spins and the resulting spin is coupled with the orbital angular momentum to give the total angular momentum of the meson. We favour this scheme in contrast to that chosen in [20] for the baryon mass problem since it facilitates the discussion of the discrete symmetries $C\,P$ and P in the parity symmetrical version of the theory.

For simplicity we pass to the rest frame of the two particle system: $J=(m,0,0,0)$. Then we are left with an ordinary nonrelativistic problem and get from (2.3c):

$$\sum_{i=1}^{3}\,(\sigma_i\otimes 1+1\otimes\sigma_i^{A}+L_i(x)+L_i(y))^{2} \cdot\varphi^{FA}(x,y) = j(j+1)\;\varphi^{FA}(x,y). \tag{3.18}$$

j denotes the total angular momentum, L_i are the components of the orbital angular momentum operators, $\boldsymbol{\sigma}^{A}$ is defined by $\boldsymbol{\sigma}^{A} = -\,\boldsymbol{\sigma}^{T}$.

We first couple the two individual spins. The eigenfunctions of the complet system are determined by the equation

$$\begin{aligned} S^2\,\varphi^{FA} &= (\boldsymbol{\sigma}\otimes 1+1\otimes\boldsymbol{\sigma}^{A})^{2}\,\varphi^{FA} \\ &= \tfrac{1}{2}\,(3+\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}^{A})\,\varphi^{FA} \\ &= s(s+1)\;\varphi^{FA} \end{aligned} \tag{3.19}$$

which follows from (3.18) by disregarding the orbital angular momentum parts. (3.19) is identical with the isospin condition (3.17), hence it follows: The eigenstates of (3.19) are

$$\begin{aligned} \sigma_0 = 1 \quad &\text{for} \quad s=0 \;\;\text{(singlet)}, \\ (\sigma_1,\sigma_2,\sigma_3) \quad &\text{for} \quad s=1 \;\;\text{(triplet)}. \end{aligned}$$

The σ-matrices are eigenstates to S^2 but not to the spinprojection s_3. Therefore the wavefunction $\varphi^{FA}_{\mu} := \mathrm{Tr}(\sigma_\mu\,\varphi^{FA})$ for definite spin ($s=0$ for $\mu=0$, $s=1$ for $\mu=1,\,2,\,3$) has not definite spinprojection. This can be achieved by transforming $\varphi^{\mu}(p)$ into the spherical basis [19, 21]. For an arbitrary four vector p^{μ} the transformation is defined by

$$\hat{p}^{\,r r_3} = u^{+\,r r_3}_{\mu}\;p^{\mu} \tag{3.20}$$

with the unitary transformation matrix

$$u^{+} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \end{pmatrix}. \tag{3.21}$$

$(r\,r_3)$ takes on the values (00, 11, 10, 1−1). $\hat{p}^{\,r r_3}$ can be written in terms of spherical harmonics:

$$\hat{p}^{\,r r_3} = \bar{p}^{\,r}\,\sqrt{\frac{4\pi}{2r+1}}\;y^{\times}_{r r_3}(\vartheta,\varphi) \tag{3.22}$$

with $\bar{p}^0 = p^0$, $\bar{p}^1 = |\boldsymbol{p}|$.

In the spherical basis the fundamental tensor is defined by:

$$g^{r r_3,\,s s_3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{3.23}$$

It can be used to raise respectively to lower the indices. The ε-tensor in the spherical basis is simply multiplied by i because of $\det U^{+} = i$.

By applying (3.20) to φ^{μ} we get a wavefunction for definite spin and spin projection:

$$\varphi^{s s_3} = u^{+\,s s_3}_{\mu}\;\varphi^{\mu}\,. \tag{3.24}$$

To determine the angular momentum dependence of $\varphi(x_1, x_2)$ we have to examine the relation

$$\sum_{i=1}^{3} (L_i(x) + L_i(y))^2 \varphi(x,y) = l(l+1)\, \varphi(x,y). \quad 3.25)$$

Fourier-Transformation of (3.25) and separation of the center-of-mass motion gives in the rest system $J = (m, 0, 0, 0)$:

$$L^2 \tilde{\varphi}(q) = \sum_{i=1}^{3} L_i^2(q)\, \tilde{\varphi}(q) = l(l+1)\, \tilde{\varphi}(q). \quad (3.26)$$

We are now in position to construct from φ^{ss_3} the wave function $\varphi^{sl}_{jj_3}$, which carries definite total angular momentum j. In the first step we expand φ^{ss_3} in eigenfunctions of L^2, that is in spherical harmonics:

$$\varphi^{ss_3}(q) = \sum_{l l_3} \varphi^{ss_3, l_3}(q_0 \,|\, \boldsymbol{q}\,|)\, y_{l l_3}(\Omega_q), \quad \Omega_q = (\vartheta, \varphi). \quad (3.27)$$

For φ^{ss_3, l_3} the Clebsch-Gordan expansion holds:

$$\varphi^{ss_3, l_3} = \sum_{j j_3} c(s\, l\, j, s_3\, l_3\, j_3)\, \varphi^{sl}_{jj_3}. \quad (3.28)$$

For the notation of the Clebsch-Gordan coefficients see [23]. Insertion of (3.28) into (3.27) gives:

$$\varphi^{ss_3}(q) = \sum_{\substack{l l_3 \\ j j_3}} c(s\, l\, j, s_3\, l_3\, j_3)\, y_{l l_3}(\Omega_q)\, \varphi^{sl}_{jj_3}(q_0 \,|\, \boldsymbol{q}\,|). \quad (3.29)$$

By application of the orthogonality relations for Clebsch-Gordan coefficients and spherical harmonics we get the desired wavefunction with definite total angular momentum:

$$\varphi^{sl}_{jj_3}(q_0 \,|\, \boldsymbol{q}\,|) = \sum_{s_3 l_3} c(s\, l\, j, s_3\, l_3\, j_3) \cdot \int \mathrm{d}\Omega_q\, y^{\times}_{l l_3}(\Omega_q)\, \varphi^{ss_3}(q). \quad (3.30)$$

4. Partial Wave Analysis

a) We expand $\varphi_{\alpha\beta}$ in baryon number eigenfunctions Λ^{AB}

$$\varphi = \sum_{\mathrm{AB}} \Lambda^{\mathrm{AB}}\, \varphi^{\mathrm{AB}}. \quad (4.1)$$

Application of $\Lambda^{+\mathrm{FA}}$ and $\Lambda^{+\mathrm{AF}}$ yields by means of (3.15) a coupled system for φ^{FA} and φ^{AF}. These functions are related by

$$\varphi^{\mathrm{FA}}_{\alpha\beta}(q) = -\varphi^{\mathrm{AF}}_{\beta\alpha}(-q). \quad (4.2)$$

(4.2) is the generalized Paule principle. By this φ^{AF} can be eliminated in the equation for φ^{FA}, giving a single equation for φ^{FA}.

In the next step, we expand φ^{FA} in isospin, and spin eigenfunctions

$$\varphi^{\mathrm{FA}} = \tau^{\nu}\, \bar{\sigma}^{\mu}\, \varphi^{\mathrm{FA}}_{\nu\mu}, \quad \nu, \mu = 0, 1, 2, 3. \quad (4.3)$$

By applying the orthonormality relations for the τ- and σ-matrices we get an equation for definite baryon number, isospin, and spin:

$$\begin{aligned} m^2 \varphi^{\mathrm{FA}(\binom{0}{1})\varkappa}(q) = {} & \frac{8\, l^4}{(2\pi)^8} \binom{9}{1} [G_{\mu}^{\,h}(q_-)\, F_{\lambda}(q_+) + G_{\lambda}^{\,h}(q_+)\, F_{\mu}(q_-)]\, M_{\nu h \tau}(J)\ \bar{R}^{\varkappa\mu\delta\lambda}\, \hat{R}^{\tau}{}_{\nu'}{}^{\nu}{}_{\delta} \int \mathrm{d}p\, \varphi^{\mathrm{FA}(\binom{0}{1})\nu'}(p) \\ & - \frac{16\, l^4}{(2\pi)^8} \binom{3}{2} [G_{\mu}^{\,h}(q_+)\, F_{\tau}(q_-) + G_{\tau}^{\,h}(q_-)\, F_{\mu}(q_+)]\, [\delta_{\nu'}{}^{\nu}\, R^{\mu\lambda\tau\varkappa} + \delta_{\nu'}{}^{\lambda}\, R^{\mu\nu\tau\varkappa}] \int \mathrm{d}p\, M_{\lambda h \nu}(q-p)\, \varphi^{\mathrm{FA}(\binom{0}{1})\nu'}(p) \\ & + 8 \frac{l^4}{(2\pi)^8} \binom{3}{1} [G_{\mu}^{\,h}(q_+)\, F_{\tau}(q_-) + G_{\tau}^{\,h}(q_-)\, F_{\mu}(q_+)]\, R^{\mu}{}_{\nu'}{}^{\tau\varkappa} \int \mathrm{d}p\, M_{\lambda h}{}^{\lambda}(q+p)\, \varphi^{\mathrm{FA}(\binom{0}{1})\nu'}(p) \end{aligned} \quad (4.4)$$

with the following definitions:

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= i\, \varepsilon_{\alpha\beta\gamma\delta} + g_{\alpha\beta}\, g_{\gamma\delta} - g_{\alpha\gamma}\, g_{\beta\delta} + g_{\alpha\delta}\, g_{\beta\gamma}, \\ R_{\alpha\beta\gamma\delta} &= -i\, \varepsilon_{\alpha\beta\gamma\delta} + g_{\alpha\beta}\, g_{\gamma\delta} - g_{\alpha\gamma}\, g_{\beta\delta} + g_{\alpha\delta}\, g_{\beta\gamma}, \\ R_{\alpha\beta\gamma\delta} &= \phantom{-i\, \varepsilon_{\alpha\beta\gamma\delta}} + g_{\alpha\beta}\, g_{\gamma\delta} - g_{\alpha\gamma}\, g_{\beta\delta} + g_{\alpha\delta}\, g_{\beta\gamma}, \\ F_{\lambda}(q) &= i\, \varkappa^4\, q_{\lambda}/(q^2)^2\, (q^2 - \varkappa^2), \quad G_{\mu}^{\,h}(q) = q^h\, q_{\mu}/q^2. \end{aligned} \quad (4.5)$$

"FA" refers to the baryon number which is 0 in our case. This index will be suppressed in the following. $\binom{0}{1}$ denotes the isospin of the state. The equations for different isospins vary only by numerical factors which are given in brackets. $\varkappa$ refers to the (algebraic) spin which still must be coupled to the orbital angular momentum. In the rest system $J = (m, 0, 0, 0)$, $\varkappa = 0$ belongs to $s = 0$ and $\varkappa = 1, 2, 3$ belong to $s = 1$. In (4.4) one can clearly see the different contributions: The first term is a local term which is essentially equivalent to the lowest Tamm-Dancoff approximation. The second term is a normal Bethe-Salpeter kernel in a ladder like approximation, and the third one is an exchange type kernel.

Besides the above mentioned symmetries we have as a discrete symmetry the CP-invariance. As is well known the eigenvalue of CP for a two particle system is given by $(-1)^{s+1}$ (see [22]). Hence in (4.4) $s = 0$ should decouple from $s = 1$ to get CP-eigenfunctions. This is achieved by multiplying the right hand side by $\frac{1}{2}(1 + (-1)^{\varkappa+\nu'})$.

b) In the next step we have to carry out the partial wave expansion of Eq. (4.4) in the second and third term. This is not necessary for the first local term since there no orbital angular momentum occurs. This has been done for similar equations in [20] by using a method which is far too complicated and by no means adequate. It requires the use of higher $3\,n$ j-Symbols and elaborate graphical techniques [21]. Our method uses 6 j-Symbols in one term only and Clebsch-Gordan coefficients in the other. In an obvious shorthand notation Eq. (4.4) reads:

$$m^2\,\varphi^{\eta}(q) = \int \mathrm{d}p\, K^{\eta}{}_{\lambda}(q,p,J)\;\varphi^{\lambda}(p). \quad (4.6)$$

(4.6) can be cast into the form

$$m^2\,\varphi^{\eta}(q) = \int \mathrm{d}p \sum_{ij} a^{\eta}{}_i\, b_{j,\lambda}\, k^{ij}(q,p,J)\;\varphi^{\lambda}(p). \quad (4.7)$$

$a_i{}^{\eta}\, b_{j,\lambda}$ runs through all possible second rank tensors which can be formed from the four vectors q, p and J together with the tensors $g^{\mu\nu}$ and $\varepsilon^{\mu\nu\varrho\lambda}$. Application of the transformation (3.21) yields:

$$m^2\,\varphi^{ss_3}(q) = \int \mathrm{d}p \sum_{\substack{ij \\ \lambda\lambda_3}} a_i{}^{ss_3}\, b_{j,\lambda\lambda_3}\, k^{ij}(q,p,J)\;\varphi^{\lambda\lambda_3}(p). \quad (4.8)$$

We expand the kernels in spherical harmonics:

$$k^{ij}(q,p,J) = \sum_{LL_3} k_L^{ij}\,(q_0,q_1,p_0,p_1,J) \cdot y_{LL_3}(\Omega_q)\; y^{\times}_{LL_3}(\Omega_p) \quad (4.9)$$

with $$q_1 := |\boldsymbol{q}|.$$

Then from (4.8) follows with (3.30) and (3.29) the equation for definite total angular momentum:

$$m^2\,\varphi^{sl}_{jj_3}(q_0,q_1) = \sum_{\substack{ij \\ LL_3}} \sum_{\substack{s_3 l_3 \\ \lambda\lambda_3}} c(s\,l\,j, s_3\,l_3\,j_3) \int \mathrm{d}\Omega_q\,\mathrm{d}p\; a_i{}^{ss_3}\, b_{j,\lambda\lambda_3} \cdot k_L^{ij}\,(q_0,q_1,p_0,p_1)\; y_{LL_3}(\Omega_q)\; y^{\times}_{LL_3}(\Omega_p) \quad (4.10)$$
$$\sum_{\substack{l'l_3' \\ j'j_3'}} c(\lambda\, l'\, j', \lambda_3'\, j_3')\; y_{l'l_3'}(\Omega_p)\; \varphi^{\lambda l'}_{j'j_3'}(p_0,p_1).$$

The expression (4.10) is simplified considerably if the angular integrations are performed. This can be done without specifying the angular momenta involved for all combinations $a_i\, b_j$. The relevant formulas are listed in Appendix I. Here we are interested only in vector mesons, i. e. in $j=1$. Then we have four possible combinations for s and l, namely $(s\,l) = (01)$, (10), (11), (12). As stated above, $s=0$ decouples from $s=1$ due to CP-invariance. In a parity symmetrical version, (11) would additionally decouple from (10) and (12). As we are interested only in the existence problem for vector mesons in nonlinear spinor theory, we restrict ourselves to the most simple case $s=0$, $l=1$. This choice corresponds to a fermion-antifermion bound state in a $^1P_1$ state. The particle with isospin 1 is known as the B-meson with $m=1235$ MeV [22], the isospin 0 particle has not been found experimentally so far.

For a state with $l\neq 0$ like $^1P_1$, the local graph in (3.23) vanishes. With the help of (4.4), (4.10), and Apendix I, the resulting equation can be put down immediately. Because the resulting expression is rather lengthy we wil restrict ourselves in the next section to an approximation of it.

5. The Fredholm Approximation

a) After angular momentum decomposition the eigenvalue equation is of a form which could be treated by the standard methods used for the numerical solution of Bethe-Salpeter equations with a simpler interaction kernel [24]. A detailed analysis of the equations will be left to a subsequent paper. Here we will make only a first rough approximation.

As was done in [7], we treat the resulting eigenvalue equation for the $^1P_1$ state in the first Fredholm approximation. Certainly, the integral kernel is not of the Fredholm type. Especially the condition of square integrability required for the convergence of the Fredholm series is not fulfilled. But investigations on simpler problems like the anharmonical oscillator [25] show that the series converges for a much wider class of integral kernels. Hence we assume that our kernel is of such a type. As we did explicitly perform the angular decomposition contrary to [7], we are not forced to shift the mass of the regularizing dipole in (3.5) to a finite value in order to get a nice behaviour on the light cone.

It is difficult to say something about the accuracy of the aproximation. In the case of the scalar-scalar Bethe-Salpeter equation in the ladder approximation, the eigenvalues calculated from the Fredholm approximation differ from the correct oney by a factor which varies slowly with the coupling constant. In particular, the Fredholm approximation leaves the norm unchanged, which is essentially determined by the sign of the derivative of the coupling constant, which is taken to be a function of the meson mass looked for.

Therefore one may hope the Fredholm approximation to be suited for the problem of existence of vector mesons in nonlinear spinor theory, whereas

the values of the calculated masses may be rather poor.

The eigenvalue equation consists of two terms. In one term, the kernel $k(q,p)$ depends essentially on the difference $q-p$ of the variables, in the other term on the sum $q+p$. From the fact that we deal with an eigenstate of CP the following symmetry relation can be derived:

$$\varphi^{sl}_{jj_3}(q_0,q_1)=\varphi^{sl}_{jj_3}(-q_0,q_1). \qquad (5.1)$$

By means of this relation the exchange term can be handled in the same way as the normal one.

In the first Fredholm approximation, the eigenvalue equation reads explicitly with

$$y=2(q_1^2/\varkappa)(z-1):$$

$$\begin{aligned} m^2=&\binom{3}{2}\frac{48\,l^4}{(2\pi)^5}\int_{-\infty}^{\infty}\mathrm{d}q_0\int_0^{\infty}\mathrm{d}q_1\,q_1^4\int_{-1}^{1}\\ &\cdot\mathrm{d}z\,z(z-1)\,g(q_-)\,f(q_+)\\ &\cdot\{q_0^2[3A^{(2)}(y)+A^{(3)}(y)]-m\,q_0[A^{(2)}(y)\\ &+\tfrac{1}{2}A^{(3)}(y)]+(q_1^2-\tfrac{1}{4}m^2)\,A^{(2)}(y)\} \qquad (5.2)\\ &-\binom{3}{1}\frac{36\,l^4}{(2\pi)^5}\int_{-\infty}^{\infty}\mathrm{d}q_0\int_0^{\infty}\mathrm{d}q_1\,q_1^4\\ &\int_{-1}^{1}\cdot\mathrm{d}z\,z(z-1)\,g(q_-)\,f(q_+)\\ &\cdot\{(q_0^2+q_1^2-\tfrac{1}{4}m^2)\,A^{(3)}(y)\}. \end{aligned}$$

In (5.2) we have used the definitions

$$g(q_-)=\frac{1}{(q_0-\frac{1}{2}m)^2-q_1^2}, \qquad (5.3)$$

$$f(q_+)=\frac{i\varkappa^4}{[(q_0+\frac{1}{2}m)^2-q_1^2]^2\,[(q_0+\frac{1}{2}m)^2-q_1^2-\varkappa^2]}.$$

Due to the degeneracy with respect to j_3, a factor $3=2j+1$ has been introduced in (5.2).

The integrals occuring in (5.2) can be evaluated explicitly. The details of the procedure can be found in Appendix II. With $\lambda=m^2/\varkappa^2$ the result has the form

$$1+\left(\frac{\varkappa l}{2\pi}\right)^4\frac{1}{16}\left\{\binom{3}{2}q_1(\lambda)+\binom{3}{1}q_2(\lambda)\right\}=0. \qquad (5.4)$$

The functions $q_1(\lambda)$ and $q_2(\lambda)$ are defined by (II.8) and (II.9).. A graphical representation of the curly bracket is given in Figure 1.

Because of the mass zero poles in $g(q)$ and $f(q)$, there are a lot of unphysical cuts in the functions $q_i(\lambda)$ starting at $\lambda=0$ and $\lambda=1$. We do not want to go into a detailed discussion of the complicated structure of the Rieman surfaces [26] but use the simple prescription to take the real part of the functions. This has been observed by the calculations in Appendix II. The physical cut starts at $\lambda=4$, and solutions lying beyond this threshold are not regarded as true bound states.

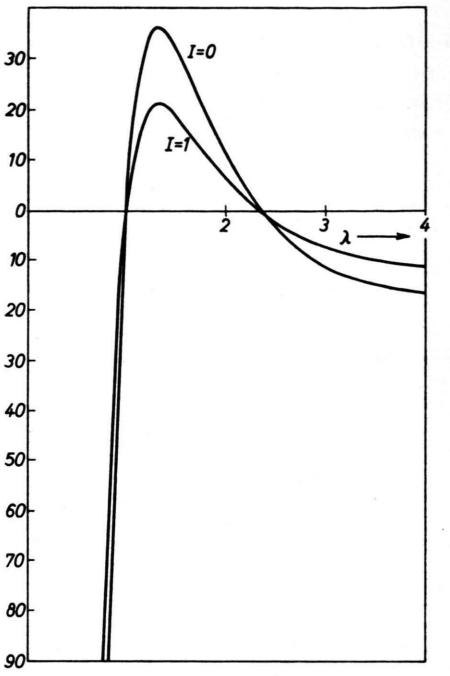


Fig. 1.

The decision whether a solution corresponds to a ghost state or to a physical particle is derived by the calculation of the norm of the state. Following the program developed in the introduction this should be done with help of the scalar product in functional space. Such a scalar product has been proposed in [27]. But until now there exist no explicit calculation of the norm of an interacting particle. Therefore we use here the conventional Bethe-Salpeter normalization procedure which is applied to the nonlinear spinor theory in [28]. There it is shown that the norm is related to the derivative of the corresponding function $q(\lambda)$. For solution with spin 1, the derivative must be negative in order to give a physical state. From Fig. 1 one sees that solutions with a negative derivative exist for a large range of the coupling constant. Hence we hawe shown the existence of vector mesons in nonlinear spinor theory. The resulting mass will be given somewhat later.

b) For the local graph in (4.4) the angular decomposition can in principle be performed in the

same way as for the nonlocal graphs. But since there is no orbital angular momentum involved in this case, we can get the desired equation of the form (5.4) in a much simpler fashion. By integrating over q we get an algebraic equation:

$$m^2\,\hat{\varphi}^{(\frac{0}{1})\varkappa} = \frac{8\,l^4}{(2\pi)^8}\binom{9}{1}\,[M_\lambda{}^h{}_\mu(J) + M_\mu{}^h{}_\lambda(J)]\,M_{\tau h\nu}(J) \cdot \bar{R}^{\varkappa\mu\delta\lambda}\,\hat{R}^{\tau}{}_{\sigma}{}^{\nu}{}_{\delta}\,\hat{\varphi}^{(\frac{0}{1})\sigma} \tag{5.5}$$

with

$$\hat{\varphi}^{\varkappa} = \int \mathrm{d}p\,\varphi^{\varkappa}(p)\,. \tag{5.6}$$

This equation can be written in the following form

$$\hat{\varphi}^{(\frac{0}{1})\varkappa} = \frac{l^4}{(2\pi)^4}\binom{9}{1}\hat{\varphi}^{(\frac{0}{1})}{}_\sigma\left\{\frac{J^\varkappa J^\sigma}{J^2}\cdot 12[2A^{(2)}(J) + A^{(3)}(J)]^2 + \left(g^{\varkappa\sigma} - \frac{J^\varkappa J^\sigma}{J^2}\right)\cdot 8[A^{(2)}(J) + A^{(3)}(J)]^2\right\}. \tag{5.7}$$

By applying to (5.7) the spin projection operators $J^\varkappa J^\sigma/J^2$ for spin 0 and $g^{\varkappa\sigma} - J^\varkappa J^\sigma/J^2$ for spin 1 we get immediately the eigenvalue equations:

$$\text{Spin 0:}\quad 1 + \binom{9}{1}\frac{1}{16}\left(\frac{\varkappa l}{2\pi}\right)^4 q_0'(\lambda) = 0\,,$$

$$\text{Spin 1:}\quad 1 + \binom{9}{1}\frac{1}{16}\left(\frac{\varkappa l}{2\pi}\right)^4 q_1'(\lambda) = 0 \tag{5.8}$$

with the functions

$$q_0'(\lambda) = -\frac{4}{3}\left[\frac{1}{\lambda^2} - \frac{5}{2\lambda} + \frac{(1-\lambda)^3}{\lambda^3}\ln(1-\lambda) + \ln\lambda\right]^2$$

$$q_1'(\lambda) = -2\left[-\frac{1}{3\lambda^2} + \frac{1}{2\lambda} + \frac{1}{3} - \frac{(1-\lambda)^3(1+\lambda)}{3\lambda^3}\ln(1-\lambda) + (\tfrac{2}{3} - \tfrac{1}{3}\lambda)\ln\lambda\right]^2. \tag{5.9}$$

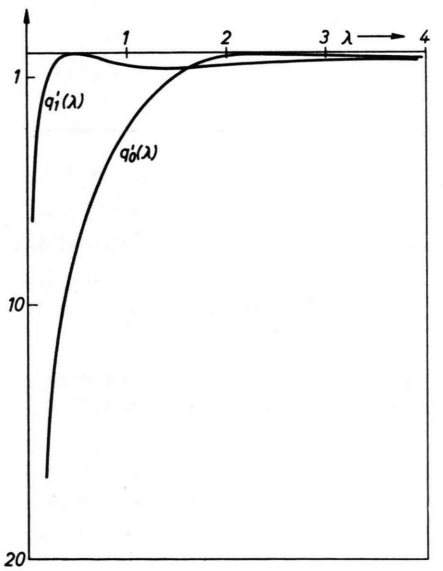


Fig. 2.

The functions q_0' and q_1' are shown in Figure 2. It is seen that the function q_1' has also a negative derivative in a small interval. But this lies in the neighbourhood of the axis yielding a coupling constant far too large. Furthermore it is doubtful at all whether the solution can be interpreted as a real particle since the function q_1' is essentially the square of the corresponding function calculated in [14] for a linearized version of Eq. (2.15) which changes sign on the positive axis and behaves like Heisenberg's $q_1(\lambda)$ and therefore the arguments in [1] for not having vector mesons do hold. This is certainly not the case for the nonlocal graph.

c) To get definite masses for the particles out of the Eqs. (4.4), (4.8) one has to fix the value of the coupling constant l^2. This is done by means of the nucleon eigenvalue Equation (3.1): l^2 is chosen in such a way that the resulting nucleon mass is identical with the input mass in the propagator. Explicitly, Eq. (3.1) reads in momentum space

$$m^2\,\varphi_\varepsilon(J) = \frac{18}{(2\pi)^8}\,G^h_{\varepsilon\varrho}(J)\,V^{\varrho\lambda\mu\nu}\,V^{\alpha\beta\gamma\delta}\,\varphi_\delta(J) \cdot \int \mathrm{d}p\,\mathrm{d}q\,G_{h\lambda\alpha}(q)\,F_{\mu\beta}(p)\,F_{\nu\gamma}(J-p-q). \tag{5.10}$$

Since the external Green function carries the total momentum J, (5.10) can be written also:

$$J^4\,\varphi_\varepsilon(J) = \frac{18}{(2\pi)^8}\,J_h\,J_l\,G^h_{\varepsilon\varrho}(J)\,V^{\varrho\lambda\mu\nu}\,V^{\alpha\beta\gamma\delta}\,\varphi_\delta(J) \cdot \int \mathrm{d}p\,\mathrm{d}q\,G^l_{\lambda\alpha}(q)\,F_{\mu\beta}(p)\,F_{\nu\gamma}(J-p-q). \tag{5.11}$$

In this form the equation is identical with a corresponding one in [14] which was derived from a linearized form of (2.15). Hence we can take the result from [14] and get with $\lambda = J^2/\varkappa^2$:

$$\left(1 + \tfrac{3}{2}\,(\varkappa l/2\pi)^4\,L(\lambda)\right)\varphi_\alpha(J) = 0\,. \tag{5.12}$$

The function $L(\lambda)$ is defined in [14]. The selfconsistency condition now requires:

$$(\varkappa l/2\pi)^4 = -2/3\,L(1) \tag{5.13}$$

and with

$$L(1) = \frac{5}{4} - \frac{\pi^2}{12} \approx 0.42753 \tag{5.14}$$

we get for the coupling constant

$$(\varkappa l/2\pi)^4 = -1.559\,. \tag{5.15}$$

This would mean an imaginary value of l^2 in the equation for the field operators. In the conventional

approach to field theory this would not make sense, but if one assumes the functional Eq. (2.15) to be the definition of the theory, this is not so certain.

For a comparison, we give the coupling constant obtained by the Tamm-Dancoff-approximation [1]:

$$(\varkappa l/2\pi)^4 = 1.069\,. \tag{5.16}$$

If we identify $\varkappa$ with the mass of the proton: $\varkappa = 939$ MeV we get the masses of the mesons in MeV. They are listed in the table below for the two values of coupling constant (5.15), (5.16). Only solutions corresponding to physical particles are given:

State	name	(m) for (5.15)	(m) for (5.16)	experim.
$^1s_0, I=0$	η	—	1.26 (1055 MeV)	549 MeV
$^1s_0, I=1$	π	—	0.21 (429 MeV)	137 MeV
$^1p_1, I=0$	?	2.02 (1335 MeV)	3.47 (1749 MeV)	?
$^1p_1, I=1$	B	1.84 (1274 MeV)	—	1235 MeV

The computed masses are of the correct order of magnitude. But before one can make a detailed comparison with the experiment, one should make a better approximation than the Fredholm approximation.

We notice, that for the coupling constant (5.16) no eigenvalue for the B-meson exists, because the function $q(\lambda) = \{\binom{3}{2} q_1(\lambda) + \binom{3}{1} q_2(\lambda)\}$ tends to zero as λ tends to infinity. From the conventional point of view one may argue that only the values of q which correspond to positive values of $(\varkappa l/2\pi)^4$, that is the part of the graph for $q(\lambda)$ below the axis, can serve as candidates for the B-meson. In this region, $(\varkappa l/2\pi)^4$ varies between ∞ and 1.6 approximately. This corresponds to mass values between 1450 and 2000 MeV, which are stil in the correct order of magnitude.

Acknowledgment

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Appendix I

In this Appendix, we list the formulas needed for the explicit evaluation of (3.38) for the different possible tensors $a_i^\eta\, b_{\lambda j}$. The proof of the formulas is straightforward and uses only the well-known properties of spherical harmonics. Besides Clebsch-Gordan coefficients, in one term there appears the 6 j-symbol $\begin{Bmatrix} s & l & j \\ \lambda & l' & j' \end{Bmatrix}$. The notation is self-explanatory for a great deal: First we give the tensor $a^\eta\, b_\lambda$ in the Cartesian basis, and then we give the result contributing to Equation (3.38). An integration over p_0 and p_1 on the right hand side is to be understood. $k_l = k_l(q_0, q_1, p_0, p_1)$ denotes the kernel, and $[L_1 \dots L_n] \equiv (2L_1+1) \dots (2L_n+1)$.

$$q^\eta\, q_\lambda \;:\; \varphi^{sl}_{jj_3} = \sum_{\lambda L} \frac{\sqrt{[l,L]}}{[j]}\, c(s\,l\,j,0\,0\,0)\; c(\lambda\,L\,j,0\,0\,0)\; k_L\, \bar{q}^s\, \bar{q}_\lambda\, \varphi^{\lambda l}_{jj_3}$$

$$q^\eta\, p_\lambda \;:\; \varphi^{sl}_{jj_3} = \sum_{\lambda L} \frac{\sqrt{[l,L]}}{[j]}\, c(s\,l\,j,0\,0\,0)\; c(\lambda\,L\,j,0\,0\,0)\; k_j\, \bar{q}^s\, \bar{p}_\lambda\, \varphi^{\lambda l}_{jj_3}$$

$$p^\eta\, q_\lambda \;:\; \varphi^{sl}_{jj_3} = \sum_{\lambda l' L} [L]\; c(L\,\lambda\,l,0\,0\,0)\; c(L\,s\,l',0\,0\,0) \begin{Bmatrix} s & l & j \\ \lambda & l' & L \end{Bmatrix} k_L\, \bar{p}^s\, \bar{q}_\lambda\, \varphi^{\lambda L}_{jj_3}$$

$$p^\eta\, p_\lambda \;:\; \varphi^{sl}_{jj_3} = \sum_{\lambda L} \frac{\sqrt{[l,L]}}{[j]}\, c(s\,l\,j,0\,0\,0)\; c(\lambda\,L\,j,0\,0\,0)\; k_l\, \bar{p}^s\, \bar{p}_\lambda\, \varphi^{\lambda L}_{jj_3}$$

$$J^\eta\, q_\lambda \;:\; \varphi^{sl}_{jj_3} = \delta_{s0}\, \delta_{lj} \sum_{\lambda L} \sqrt{\frac{[L]}{[j]}}\, c(\lambda\,l\,j,0\,0\,0)\; k_L\, m\, \bar{q}_\lambda\, \varphi^{\lambda L}_{jj_3}$$

$$q^\eta\, J_\lambda \;:\; \varphi^{sl}_{jj_3} = \sqrt{\frac{[L]}{[j]}}\; c(s\,l\,j,0\,0\,0)\; k_j\, m\, \bar{q}^s\, \varphi^{0j}_{jj_3}$$

$$J^\eta\, p_\lambda \;:\; \varphi^{sl}_{jj_3} = \delta_{s0}\, \delta_{lj} \sum_{\lambda L} \sqrt{\frac{[L]}{[j]}}\, c(\lambda\,L\,j,0\,0\,0)\; k_j\, m\, \bar{p}_\lambda\, \varphi^{\lambda L}_{jj_3}$$

$$p^\eta\, J_\lambda \;:\; \varphi^{sl}_{jj_3} = \sqrt{\frac{[l]}{[j]}}\; c(s\,l\,j,0\,0\,0)\; k_l\, m\, \bar{p}^s\, \varphi^{0j}_{jj_3}$$

$$J^\eta\, J_\lambda \;:\; \varphi^{sl}_{jj_3} = \delta_{s0}\, \delta_{lj}\, k_j\, m^2\, \varphi^{0j}_{jj_3}$$

$$\delta_\lambda{}^\eta \;:\; \varphi^{sl}_{jj_3} = k_l\, \varphi^{sl}_{jj_3}$$

Additionally, there are some terms containing the ε-tensor, which are more complex. But as they vanish for $s=0$, they are not given here.

Appendix II

In this Appendix, the explicit integration of the Fredholm approximation is carried out, following the procedure described in [19]. The mehod is exemplified by the first term of (4.2), the second term works by completely the same pattern.

We begin with the q_0-integration. The simplest way to do this is achieved by introducing spectral functions [16] and perform the integration by means of the $i\varepsilon$-prescription and the residue theorem. Then we get from (5.2) neglecting the isospin factors with $q=q_1$ and $y=2(q^2/\varkappa^2)(z-1)$:

$$
\begin{aligned}
m^2 = \frac{6\,l^4}{(2\pi)^4}\int_0^\infty \mathrm{d}q\, q^4 \int_{-1}^{1} \mathrm{d}z\, z(z-1)\Big\{ & -\frac{m^2\varkappa^2}{q^3(4q^2-m^2)}\left(2A^{(2)}(y)+\tfrac{1}{2}A^{(3)}(y)\right) \\
& + \left[\frac{4(m^2+q^2)}{m^2-(q+\sqrt{q^2+\varkappa^2})^2}\cdot\frac{1}{\sqrt{q^2+\varkappa^2}} - \frac{8q}{m^2-(q+\sqrt{q^2+\varkappa^2})^2} - \frac{12\sqrt{q^2+\varkappa^2}}{m^2-(q+\sqrt{q^2+\varkappa^2})^2} - \frac{4}{q}\right] A^{(2)}(y) \\
& + \left[\frac{2m^2}{m^2-(q+\sqrt{q^2+\varkappa^2})^2}\cdot\frac{1}{\sqrt{q^2+\varkappa^2}} - \frac{4(q+\sqrt{q^2+\varkappa^2})}{m^2-(q+\sqrt{q^2+\varkappa^2})^2} - \frac{2}{q}+\frac{1}{2q^3}\right] A^{(3)}(y)\Big\}.
\end{aligned}
\tag{II.1}
$$

In (II.1), only $A^{(2)}$ and $A^{(3)}$ depend on the angular variable z. Therefore, we next perform the angular integration. We get with $\eta=2(q^2/\varkappa^2)$.

$$
\begin{aligned}
\hat{A}^{(2)}(\eta) &:= \int_{-1}^{1}\mathrm{d}z(z-1)\, z\, A^{(2)}(\eta(z-1)) \\
&= \frac{\varkappa^2}{2}\left[\left(\frac{31}{144\eta^3}+\frac{7}{18\eta^2}+\frac{2}{9}+\frac{1}{9\eta}\right)\ln(1+2\eta) + (-\tfrac{2}{9}-\tfrac{1}{9}\eta)\ln 2\eta\right. \\
&\quad \left. -\frac{1}{6\eta^2}\mathfrak{L}_2(-2\eta)+\frac{1}{9}-\frac{13}{36\eta}+\frac{1}{12\eta^2}\right]
\end{aligned}
\tag{II.2}
$$

$$
\begin{aligned}
\hat{A}^{(3)}(\eta) &:= \int_{-1}^{1}\mathrm{d}z\, z(z-1)\, A^{(3)}(\eta(z-1)) \\
&= \frac{\varkappa^2}{2}\left[\left(-\frac{1}{24\eta^3}-\frac{1}{36\eta}-\frac{1}{3}-\frac{2}{9}\eta\right)\ln(1+2\eta) + (\tfrac{1}{3}+\tfrac{2}{9}\eta)\ln 2\eta\right. \\
&\quad \left. -\frac{1}{6\eta^2}\mathfrak{L}_2(-2\eta)+\frac{1}{9}-\frac{13}{36\eta}+\frac{1}{12\eta^2}\right]
\end{aligned}
\tag{II.3}
$$

$\mathfrak{L}_2(x)$ denotes the dilogarithm. See [29] and Appendix III.

To perform the q-integration in the first term of (II.1), we represent the functions $A^{(2)}$ and $A^{(3)}$ in the form of parameter integrals:

$$
\begin{aligned}
\hat{A}^{(2)}(\eta) &= -\frac{\varkappa^2}{2}\int_0^1 \mathrm{d}x\left[\frac{1}{18}\frac{x-4}{x+2\eta}+\frac{23x^2-26x}{18(1+2\eta x)}+\frac{1}{12}\int_{-1}^{1}\mathrm{d}z\, z(z-1)\frac{4x-x^2}{1-\eta x(z-1)}\right], \\
\hat{A}^{(3)}(\eta) &= \frac{\varkappa^2}{2}\int_0^1 \mathrm{d}x\left[\frac{1}{9}\frac{x-3}{x+2\eta}+\frac{9x^2-11x}{9(1+2\eta x)}+\frac{1}{6}\int_{-1}^{1}\mathrm{d}z\, z(z-1)\frac{x}{1-\eta x(z-1)}\right].
\end{aligned}
\tag{II.4}
$$

With this substitution the q-integration is easily carried out. In the other terms of (II.1) we substitute

$$
q=\frac{\varkappa}{2}\frac{x}{\sqrt{1-x}}\,. \tag{II.5}
$$

With this substitution the square roots disappear. With $\lambda = m^2/\varkappa^2$ we finally get:

$$
\begin{aligned}
m^2 = {} & \frac{3}{4}\left(\frac{\varkappa l}{2\pi}\right)^4 \int_0^1 \mathrm{d}x\, \frac{x^3}{1-x}\,\frac{\lambda-1}{1-\lambda(1-x)}\,\hat{A}^{(2)}\left(\frac{x^2}{2(1-x)}\right) \\
& + \frac{3}{16}\left(\frac{\varkappa l}{2\pi}\right)^4 \int_0^1 \mathrm{d}x \left[\frac{(\lambda-1)^3}{\lambda^2}\,\frac{x}{(1-x)\,(1-\lambda(1-x))} + \frac{1-2\lambda}{\lambda}\,x - \frac{5\lambda^2-3\lambda+1}{\lambda^3}\,\frac{x}{1-x}\right]\tilde{A}^{(3)}\left(\frac{x^2}{2(1-x)}\right) \\
& + \frac{1}{8}\left(\frac{\varkappa l}{2\pi}\right)^4 \varkappa^2\lambda \int_0^1 \mathrm{d}x\left[\frac{1}{6}\,\frac{x-5}{x+\lambda}\ln\left(-\frac{x}{\lambda}\right) - \frac{1}{6}\,\frac{37x^2-41x}{\lambda x+1}\ln(-\lambda x)\right. \qquad \text{(II.6)} \\
& \left. + \frac{1}{2}\int_{-1}^1 \mathrm{d}z\,\frac{z(z-1)\,(7x-2x^2)}{\lambda x(z-1)-2}\ln\frac{\lambda x(z-1)}{2}\right].
\end{aligned}
$$

At the evaluation of (II.6) one meets a lot of divergent expressions. Their cancellation is achieved by passing to the integration limits in all divergent terms simultaneously. The integrals necessary for the evaluation of (II.6) are either elementary or can be found in Appendix III. The result can be written in the following form:

$$
1 + \left(\frac{l\varkappa}{2\pi}\right)^4 q_1(\lambda) = 0 \qquad \text{(II.7)}
$$

with the function $q_1(\lambda)$ defined by

$$
\begin{aligned}
q_1(\lambda) = {} & -\frac{11}{54}\pi^2\lambda - \frac{\pi}{3\sqrt{3}} - \frac{25\pi^2}{54} + \left(\frac{155}{6} - \frac{101\pi}{2\sqrt{3}} + \frac{293}{54}\pi^2 - 4F(1)\right)\frac{1}{\lambda} \\
& - \left(\frac{463}{9} + \frac{11}{6}\,\frac{\pi}{\sqrt{3}} + \frac{55}{18}\pi^2\right)\frac{1}{\lambda^2} - \left(\frac{1}{6} + \frac{\pi}{6\sqrt{3}} + \frac{59\pi^2}{18}\right)\frac{1}{\lambda^3} + \frac{\pi}{3\sqrt{3}}\,\frac{1}{\lambda^4} \\
& + \left(\frac{107\pi}{3\sqrt{3}} - \frac{83\pi^2}{27} - 8F(1)\right)\frac{1}{\lambda-1} + \left(\frac{59\pi^2}{27} + 8F(1)\right)\frac{1}{(\lambda-1)^2} \\
& + \left(-\frac{1}{3} + \frac{53}{6\lambda} - \frac{134}{3\lambda^2} + \frac{53}{6\lambda^3} - \frac{1}{3\lambda^4}\right)\ln(1-\lambda) + \left(\frac{1}{3} - \frac{15}{2\lambda} + \frac{83}{3\lambda^2}\right)\ln\lambda \qquad \text{(II.8)} \\
& + \left(\tfrac{1}{6}\lambda + \tfrac{5}{6}\right)\ln^2\lambda - \frac{1}{3}\left(\frac{56}{\lambda} - \frac{57}{\lambda^2} + \frac{59}{\lambda(\lambda-1)^2}\right)\mathfrak{L}_2(\lambda) + \frac{2(\lambda-1)^4\,(\lambda^2+3\lambda+1)}{3\lambda^5}\,\mathfrak{L}_2\left(\frac{\lambda}{\lambda-1}\right) \\
& + \frac{1}{3}\left(\lambda + 5 + \frac{55}{\lambda^2} + \frac{59}{\lambda^3}\right)\mathfrak{L}_2(1+\lambda) + \left(\frac{8}{\lambda^3} - \frac{14}{\lambda^2}\right)\ln\lambda\,\mathfrak{L}_2(-\lambda) \\
& - \left(\frac{16}{\lambda^3} - \frac{28}{\lambda^2}\right)\mathfrak{L}_3(-\lambda) + \left(\frac{14}{\lambda} - \frac{8}{(\lambda-1)^2}\right)F(\lambda) \\
& + \frac{1}{3}\left(\frac{(\lambda-1)^4\,(\lambda^2+3\lambda+1)}{\lambda^5} + \frac{56}{\lambda} + \frac{57}{\lambda^2} + \frac{59}{\lambda(\lambda-1)^2}\right)G(\lambda).
\end{aligned}
$$

Besides elementary functions in $q_1(\lambda)$ occur the dilogarithm $\mathfrak{L}_2(\lambda)$ the trilogarithm $\mathfrak{L}_3(\lambda)$ and the functions $F(\lambda)$ and $G(\lambda)$ which are defined in Appendix III.

The evaluation of the second term of (4.2) yields also a result in the form (II.7), but with another function $q_2(\lambda)$, defined by:

$$
\begin{aligned}
q_2(\lambda) = {} & -\frac{11}{36}\pi^2\lambda - \frac{\pi}{2\sqrt{3}} - \frac{\pi^2}{12} + \left(\frac{9}{4} - \frac{\pi}{4\sqrt{3}} - \frac{\pi^2}{4}\right)\frac{1}{\lambda} + \left(\frac{10}{3} + \frac{3\pi}{4\sqrt{3}} - \frac{\pi^2}{12}\right)\frac{1}{\lambda^2} \\
& - \left(\frac{1}{4} + \frac{5\pi}{4\sqrt{3}} + \frac{\pi^2}{4}\right)\frac{1}{\lambda^3} + \frac{\pi}{2\sqrt{3}}\,\frac{1}{\lambda^4} + \left(\frac{3\pi}{2\sqrt{3}} - \frac{\pi^2}{6}\right)\frac{1}{\lambda-1} + \frac{\pi^2}{6}\,\frac{1}{(\lambda-1)^2} \\
& - \left(\frac{1}{2} - \frac{21}{4\lambda} + \frac{11}{\lambda^2} - \frac{21}{4\lambda^3} + \frac{1}{2\lambda^4}\right)\ln(1-\lambda) + \left(\frac{3}{2\lambda^2} - \frac{13}{4\lambda} + \frac{1}{2}\right)\ln\lambda \qquad \text{(II.9)} \\
& + \left(\frac{3}{4} + \frac{\lambda}{4}\right)\ln^2\lambda - \frac{1}{2}\,\frac{\lambda^2+\lambda+1}{\lambda^2(\lambda-1)^2}\,\mathfrak{L}_2(\lambda) + \left(\frac{3}{2\lambda^3} + \frac{1}{2\lambda^2} + \frac{3}{2} + \frac{1}{2}\lambda\right)\mathfrak{L}_2(1+\lambda) \\
& + \frac{(\lambda-1)^3\,(\lambda^3-1)}{\lambda^5}\,\mathfrak{L}_2\left(\frac{\lambda}{\lambda-1}\right) - \frac{3}{\lambda^2}\ln\lambda\,\mathfrak{L}_2(-\lambda) + \frac{6}{\lambda^2}\,\mathfrak{L}_3(-\lambda) + \frac{3}{\lambda}F(\lambda) \\
& + \frac{\lambda^2+\lambda+1}{2\lambda^2}\left(\frac{(\lambda-1)^4}{\lambda^3} + \frac{1}{(\lambda-1)^2}\right)G(\lambda).
\end{aligned}
$$

It should be noted that the functions $q_i(\lambda)$ are finite at $\lambda = 1$ since all divergencies cancel. For $\lambda \to \infty$ they approach the axis from below.

Appendix III

a) Here we give some definitions and notations concerning the polylogarithms [29]. They are defined by:

$$\mathfrak{L}_n(z) = \int_0^z \frac{\mathfrak{L}_{n-1}(x)}{x}\,dx, \quad \mathfrak{L}_1(z) = -\ln(1-z). \tag{III.1}$$

For $|z|<1$ holds the power series expansion:

$$\mathfrak{L}_n(z) = \sum_{\nu=1}^{\infty} z^\nu/\nu^n. \tag{III.2}$$

Especially for $\mathfrak{L}_2$ and $\mathfrak{L}_3$ one gets from (IIII.1):

$$\mathfrak{L}_2(z) = -\int_0^z \frac{\ln(1-x)}{x}\,dx = \int_0^1 \frac{\ln x}{x-1/z}\,dx, \qquad \mathfrak{L}_3(z) = \int_0^z \frac{\mathfrak{L}_2(x)}{x}\,dx = \int_0^1 \frac{\ln x}{x}\ln(1-xz)\,dx. \tag{III.3}$$

For $x>1$, x real, $\mathfrak{L}_n(x)$ becomes complex. In (II.8), (II.9) it was understood that only the real part should be taken. The real part of $\mathfrak{L}_2(z)$ for complex z is given by

$$\operatorname{Re}\mathfrak{L}_2(z) = \operatorname{Re}\mathfrak{L}_2(r\,e^{i\vartheta}) \equiv \mathfrak{L}_2(r,\vartheta) = -\frac{1}{2}\int_0^r \frac{\ln(1-2x\cos\vartheta+x^2)}{x}\,dx. \tag{III.4}$$

b) Here we list some integrals related to the dilogarithm which cannot be found in the standard tables.

$$\int_0^1 \frac{\ln(1-x)}{x}\,dx = -\mathfrak{L}_2(1) = -\frac{\pi^2}{6},$$

$$\int_0^1 \frac{1}{x}\ln(1-x+x^2)\,dx = -\frac{\pi^2}{18},$$

$$\int_0^1 \frac{1}{x}\,\mathfrak{L}_2\left(-\frac{x^2}{1-x}\right)dx = -F(1) = -1.7250032\ldots,$$

$$\int_0^1 \frac{1}{x^2}\,\mathfrak{L}_2\left(-\frac{x^2}{1-x}\right)dx = \frac{\pi^2}{9} - \frac{2\pi}{\sqrt{3}},$$

$$\int_0^1 \frac{1}{x}\ln x\ln(1-x+x^2)\,dx = \tfrac{2}{3}\,\mathfrak{L}_3(1), \quad \mathfrak{L}_3(1) = 1.2020569\ldots,$$

$$\int_0^1 \frac{\ln(1-x)}{\lambda x-1}\,dx = \frac{1}{\lambda}\,\mathfrak{L}_2\left(\frac{\lambda}{\lambda-1}\right),$$

$$\int_0^1 \frac{1}{x}\ln(-\lambda x)\ln(1+\lambda x)\,dx = -\ln(-\lambda)\,\mathfrak{L}_2(-\lambda) + \mathfrak{L}_3(-\lambda),$$

$$\int_0^1 dx\,\frac{ax-b}{x+\lambda}\ln\left(-\frac{x}{\lambda}\right) = -a - a\ln(-\lambda) + (a\lambda+b)\left\{\ln\lambda\ln(1+\lambda) - \ln^2\lambda - \mathfrak{L}_2\left(-\frac{1}{\lambda}\right)\right\},$$

$$\int_0^1 \mathrm{d}x\, \frac{a\,x^2-b\,x}{1+\lambda\,x} \ln(-\lambda\,x) = \ln\lambda \left\{ \left(\frac{a}{\lambda^3}+\frac{b}{\lambda^2}\right) \ln(1+\lambda) - \frac{a}{\lambda^2} + \left(\frac{a}{2}-b\right)\frac{1}{\lambda} \right\}$$
$$+ \left(\frac{a}{\lambda^3}+\frac{b}{\lambda^2}\right) \mathfrak{L}_2(-\lambda) + \left(b-\frac{a}{4}\right)\frac{1}{\lambda},$$

$$\int_0^1 \mathrm{d}x \int_{-1}^1 \mathrm{d}z\, z(z-1)\, \frac{a\,x^2-b\,x}{\lambda(1-z)\,x+2} \ln\left(\frac{z-1}{2}\,\lambda\,x\right) = \frac{a-2\,b}{2\,\lambda} + \frac{10\,a+4\,b}{\lambda^2} - \frac{6\,a+4\,b}{\lambda^2}\ln\lambda$$
$$+ (2\,a+4\,b)\,\frac{1+\lambda}{\lambda^3}\left[\ln\lambda\ln(1+\lambda)+\mathfrak{L}_2(-\lambda)\right] + \left(\frac{4\,a}{\lambda^3}-\frac{3\,b}{\lambda^2}\right)\left[2\,\mathfrak{L}_3(-\lambda) + \ln\lambda\,\mathfrak{L}_2(-\lambda)\right].$$

c) Here the functions $F(\lambda)$ and $G(\lambda)$ are discussed. They are defined by:

$$G(\lambda) = \lambda \int_0^1 \frac{\ln(1-x+x^2)}{\lambda\,x-1}\,\mathrm{d}x, \qquad F(\lambda) = \int_0^1 \frac{1}{\lambda(1-x)-1}\,\mathfrak{L}_2\left(-\frac{x^2}{1-x}\right)\mathrm{d}x. \tag{III.5}$$

In principle, the integrals can be evaluated exactly yielding complicated expressions involving the real parts of dilogarithms and trilogarithms. So we do not give the explicit result. All important properties can be traced from the integral expressions.

In the neighbourhood of $\lambda = 1$ we have the expansions

$$G(\lambda) = \mathfrak{L}_2(\lambda) - \frac{\pi^2}{9} - \frac{\pi}{\sqrt{3}}(\lambda-1) + \ln(\lambda-1)\,(\lambda-1)^2 + O((\lambda-1)^2), \tag{III.6}$$
$$F(\lambda) = F(1) + \left(-\frac{\pi^2}{9} - \frac{2\,\pi}{\sqrt{3}} - F(1)\right)(\lambda-1) - \ln(\lambda-1)(\lambda-1)^2 + O((\lambda-1)^2).$$

For the numerical evaluation the integrals must be cast into a form in which the integrand has no singularities in the range of integration. This can be done by repeated partial integration. For the numerical integration a 48 mesh-point Gaussian quadrature formula has been used.

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